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Refining connected topological group topologies on Abelian torsion-free groups [1]

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Abstract

A technique of refining connected topological group topologies on Abelian groups is developed. It is proved that every connected separable Abelian torsion-free topological group admits a strictly finer connected separable topological group topology. © 1998 Elsevier Science B.V.

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1. Introduction

Let $\mathscr{P}$ be a topological (or topological group) property. The following general problem was considered in [2, 4, 5, 12, 13] for various properties $\mathscr{P}$.

Problem 1.1. *Let G be a topological group satisfying $\mathscr{P}$. Does there exists a strictly finer topological group topology on G still satisfying $\mathscr{P}$?*

In [4], the property $\mathscr{P}$ under consideration is pseudocompactness. It is shown there that an Abelian pseudocompact group G admits a strictly finer pseudocompact topological group (TG) topology under some additional restrictions (for example, it suffices to assume that G is a non-metrizable torsion group). A strengthening of compact TG topologies to countably compact TG topologies is considered in [2, 5]. The problem of refining locally compact group topologies was investigated in [12, 13].

The problem of refining connected topologies in the class of topological spaces was investigated in [8, 9]. The second of these articles presents a construction of a maximal

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connected Hausdorff topology on the reals. It is still an open problem whether there exists a maximal connected Tikhonov space of cardinality greater than 1.

A starting point of the present investigation is the article [10] containing a construction of a second countable connected TG topology on the additive group of reals which is strictly finer than the usual one. One more construction of a connected topological group refiniment of the topology on the reals is given in [1]. We consider connected TG topologies on Abelian groups. The main result of the paper is Theorem 4.5 which states that every separable connected Abelian torsion-free group admits a strictly finer separable connected TG topology. To prove Theorem 4.5, we present the necessary auxiliary facts in Section 4 the most important of which is Proposition 4.2 which also contains a proof of the main result.

The problem of refining connected group topologies on torsion groups will be considered in a forthcoming paper.

2. Terminology and notation

We use, respectively, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{T}$ to denote the set of non-negative integers and Abelian groups of integers, reals and the circle group. The set of positive integers is denoted by $\mathbb{N}^+$. The symbol $\mathfrak{c}$ stands for the cardinality of continuum, $\mathfrak{c} = 2^{\aleph_0}$.

The weight, density and cellularity of a space X are denoted by $w(X)$, $d(X)$ and $c(X)$, respectively. The closure of a subset S of a space X is $cl_X S$ or simply $cl\, S$. A regular closed subset of a space X is a set of the form $cl\, U$ with U open in X.

We denote the subgroup of a group G generated by a subset $D \subseteq G$ by $\langle D \rangle$. A subgroup H of a group G is called *pure* [14] if $nG \cap H = nH$ for each $n \in \mathbb{N}^+$. The cardinality of a set A is $|A|$.

In what follows, all groups are assumed to be Abelian. We consider only Hausdorff topological groups.

3. Preliminary facts and results

The following notion was introduced and studied in [7].

Definition 3.1. A topological group G is called $\aleph_0$-bounded if it can be covered by countably many translations of any neighborhood of the identity.

By Corollary 1 of [7], a topological group G is $\aleph_0$-bounded iff it is topologically isomorphic to a subgroup of a Cartesian product of second countable groups. Every separable group and every group of countable cellularity is $\aleph_0$-bounded.

Lemma 3.2. *Let G be an $\aleph_0$-bounded torsion-free group with $|G| = \lambda > \aleph_0$ and H a subgroup of G, $|H| < \lambda$. Then for every non-empty open subset U of G there exists an element $g \in U$ such that $\langle g \rangle \cap H = \{0_G\}$.*

Proof. Let U be a neighborhood of the identity in G. Since translations preserve the cardinality, we have $|U| = \lambda$. For all $m \in N^+$ and $h \in H$ put $K_{h,m} = \{g \in U\colon mg = h\}$. If the conclusion of the lemma fails then $U = \bigcup\{K_{h,m}\colon h \in H,\ m \in N^+\}$. Since, $|H \times N^+| < \lambda$, one can find that $h \in H$ and $m \in N^+$ such that $|K_{h,m}| \geq 2$. Let g_1 and g_2 be distinct elements of $K_{h,m}$. Then $mg_1 = h = mg_2$, whence $m(g_1 - g_2) = 0_G$. This contradicts our assumption that G is torsion-free. □

The same reasoning applies to prove the following slight generalization of Lemma 3.2.

Corollary 3.3. *Let $t(G)$ be the torsion-subgroup of an uncountable $\aleph_0$-bounded topological group G and suppose that $|G/tG| = |G|$. If U is a non-empty open subset of G and H is a subgroup of G with $|H| < |G|$ then there is $g \in U$ with $\langle g \rangle \cap H = \{0_G\}$.*

The result below is a special case of Theorem 3 of [15].

Lemma 3.4. *Let X be a space satisfying $w(X) \leq \mathfrak{c}$ and $c(X) \leq \aleph_0$. Then the family $RC(X)$ of all regular closed subsets of X has cardinality not greater than $\mathfrak{c}$.*

4. Main results

We prove the following theorem here. To avoid trivialities, the group G below must not be a singleton.

Theorem 4.1. *Let G be a connected Abelian torsion-free group satisfying $w(G) \leq \mathfrak{c}$ and $c(G) \leq \aleph_0$. Then G admits a strictly finer connected topological group topology satisfying the same cardinal restrictions.*

A proof of this result still requires certain auxiliary facts. We start with a proposition generalizing Claim 2.11 of [1].

Proposition 4.2. *Let G be a dense disconnected subspace of the product $X \times Y$ of connected spaces such that $p(G) = X$, where $p\colon X \times Y \to X$ is the projection. Then there exists a closed subset F of $X \times Y$ disjoint from G such that $p(F)$ has a non-empty interior. In fact, $F = cl\,O_1 \cap cl\,O_2$ for some open disjoint subsets O_1, O_2 of $X \times Y$ which cover G.*

Proof. Since G is dense in $X \times Y$ and disconnected, one can find disjoint open sets O_1 and O_2 in $X \times Y$ so that $G \subseteq O_1 \cup O_2$. Note that $X \times Y = cl\,O_1 \cup cl\,O_2$. Put $F = cl\,O_1 \cap cl\,O_2$. The sets G and F are disjoint because G is covered by O_1 and O_2.

Let $x \in X \setminus p(F)$ be an arbitrary point. Then $p^{-1}(x) \cap F = \emptyset$ and we claim that either $p^{-1}(x) \subseteq cl\,O_1$ or $p^{-1}(x) \subseteq cl\,O_2$. Indeed, otherwise the equality

$$p^{-1}(x) = (p^{-1}(x) \cap cl\,O_1) \cup (p^{-1}(x) \cap cl\,O_2)$$

would give us a partition of $p^{-1}(x)$ into two disjoint clopen subsets, which contradicts the connectedness of the space $p^{-1}(x) \cong Y$.

For a subset $O \subseteq X \times Y$, define

$$p^{\#}(O) = X \setminus p(X \times Y \setminus O).$$

It is clear that $p^{\#}(O) = \{x \in X\colon\ p^{-1}(x) \subseteq O\}$. Let us verify the following inclusion:

$$X \setminus p(F) \subseteq p^{\#}(cl\, O_1) \cup p^{\#}(cl\, O_2). \tag{1}$$

Let $x \in X \setminus p(F)$. Then either $p^{-1}(x) \subseteq cl\, O_1$ or $p^{-1}(x) \subseteq cl\, O_2$, whence either $x \in p^{\#}(cl\, O_1)$ or $x \in p^{\#}(cl\, O_2)$, which proves (1).

Since the projection p is open, both sets $p^{\#}(cl\, O_1)$ and $p^{\#}(cl\, O_2)$ are closed in X. It is easy to see that these sets are disjoint. Indeed, if $x \in p^{\#}(cl\, O_1) \cap p^{\#}(cl\, O_2)$, then $p^{-1}(x) \subseteq cl\, O_1 \cap cl\, O_2 = F$. Since $p(G) = X$, there exists $y \in Y$ with $(x, y) \in G$. But then $(x, y) \in G \cap F \neq \emptyset$, a contradiction.

Note that $p^{\#}(cl\, O_1) \neq X \neq p^{\#}(cl\, O_2)$, otherwise either $cl\, O_1 = X \times Y$ or $cl\, O_2 = X \times Y$ which is impossible. Since X is connected, we conclude that

$$X \neq p^{\#}(cl\, O_1) \cup p^{\#}(cl\, O_2).$$

Therefore, (1) implies that the non-empty open set $X \setminus (p^{\#}(cl\, O_1) \cup p^{\#}(cl\, O_2))$ is contained in $p(F)$. This proves the proposition. □

In what follows we shall say that a space X is ccc if it satisfies $c(X) \leq \aleph_0$. According to [3] a space Y is called *productively* ccc if $X \times Y$ is ccc for every ccc space X. The following result can be found in [3, Ch. 7]. We give a short proof of it for the sake of completeness.

Lemma 4.3. *A product of a ccc space and a separable space is also ccc, so every separable space is productively ccc.*

Proof. Let X be a ccc space and Y a separable one. Denote by $S = \{y_n\colon n \in N^+\}$ a countable dense subset of Y. Suppose we are given an uncountable family γ of non-empty open subsets of the product $X \times Y$. Without loss of generality, one can assume that every element $W \in \gamma$ is of the form $W = U \times V$ for some open sets $U \subseteq X$ and $V \subseteq Y$. Let $\{W_\alpha = U_\alpha \times V_\alpha\colon \alpha < \omega_1\}$ be a subfamily of γ with $W_\alpha \neq W_\beta$ whenever $\alpha \neq \beta$. Since, S is dense in Y and countable, there exists $n \in \mathbb{N}^+$ such that the set $A = \{\alpha < \omega_1\colon y_n \in V_\alpha\}$ is uncountable. The fact that X is ccc implies that $U_\alpha \cap U_\beta \neq \emptyset$ for some distinct $\alpha, \beta \in A$. Then $W_\alpha \cap W_\beta \neq \emptyset$ for these α and β. The latter means that $X \times Y$ does not contain uncountable disjoint families of open sets. □

Proof of Theorem 4.1. The idea of our proof is to construct a discontinuous homomorphism h of G to the circle group $\mathbb{T}$ in such a way that the subgroup

$$G^* = \{(x, h(x))\colon x \in G\}$$

of $G \times \mathbb{T}$ with the induced topology will be dense in $G \times \mathbb{T}$ and connected. It is clear that every dense subgroup G^* of $G \times \mathbb{T}$ satisfies the inequalities $w(G^*) \leq \mathfrak{c}$ and $c(G^*) \leq \aleph_0$ because of Lemma 4.3 and the fact that a dense subspace of a ccc space is also ccc.

The restriction of the projection $p : G \times \mathbb{T} \to G$ to G^* is a continuous injective homomorphism of G^* onto G, but $\pi = p \restriction G^*$ is not a homeomorphism (otherwise h would be continuous).

Consider the weakest topological group topology on G which makes the homomorphism $(\pi)^{-1}$: $G \to G^*$ continuous. In other words, identify the group G with G^* by means of the isomorphism π. Then, we will show that this gives us a strictly finer connected group topology on G satisfying the same cardinal restrictions.

We start with a construction of the homomorphism $h : G \to \mathbb{T}$. Denote by $\mathscr{F}$ the family of all closed subsets F of the product $G \times \mathbb{T}$ which have the form $F = cl\, O_1 \cap cl\, O_2$ for some open sets O_1, O_2 in $G \times \mathbb{T}$ and satisfy the condition $Int\ p(F) \neq \emptyset$ (Int stands for interior). Lemma 3.4 implies that $|\mathscr{F}| \leq \mathfrak{c}$. Let $\{F_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathscr{F}$. Suppose that we have constructed a set $X_\beta = \{x_\gamma : \gamma < \beta\} \subseteq G$ for some $\beta < \mathfrak{c}$. Denote by H_β the subgroup of G generated by the set X_β, $H_\beta = \langle X_\beta \rangle$. The group G is ccc, and hence is $\aleph_0$-bounded [7]. Being connected and non-trivial, G is of cardinality at least $\mathfrak{c}$. Applying Lemma 3.2, we can pick a point $x_\beta \in Int\ p(F_\beta) \setminus H_\beta$ so that

$$\langle x_\beta \rangle \cap H_\beta = \{0_H\}. \tag{2}$$

Repeat this procedure for every $\beta < \mathfrak{c}$. It gives us a set $X = \{x_\beta : \beta < \mathfrak{c}\}$. Our construction implies that $x_\beta \in p(F_\beta)$ for each $\beta < \mathfrak{c}$, so one can define a function $f : X \to \mathbb{T}$ so that $(x_\beta, f(x_\beta)) \in F_\beta$ for all $\beta < \mathfrak{c}$.

Note that the set $Gr(f) = \{(x, f(x)) : x \in X\}$ is dense in $\Pi = G \times \mathbb{T}$. Indeed, the set $F_U = cl\, U \cap cl\, U = cl\, U$ belongs to $\mathscr{F}$ for each non-empty open subset U of Π and $Gr(f)$ intersects all the sets F_U. Since the space Π is regular, we conclude that $Gr(f)$ is dense in Π.

The condition (2) of our recursive construction implies that $\langle X \rangle = \bigoplus_{x \in X} \langle x \rangle$ is a free Abelian group with the generating set X. Therefore, f extends to a homomorphism $\hat{f} : \langle X \rangle \to \mathbb{T}$. Since the group $\mathbb{T}$ is divisible, one can extend $\hat{f}$ to a homomorphism h: $G \to \mathbb{T}$ [14]. Consider the group G^* mentioned in the beginning of the proof:

$$G^* = \{(x, h(x)) : x \in G\} \subseteq \Pi.$$

The group G^* contains the set $\mathrm{Gr}(f)$, and hence is dense in Π. It remains to show that G^* is connected. Assume the contrary and apply Proposition 4.2 to find an element $F \in \mathscr{F}$ with $F \cap G^* = \emptyset$. Then $Gr(f) \cap F = \emptyset$, which contradicts the fact that $Gr(f)$ intersects all members of $\mathscr{F}$.

Thus, G^* is a dense connected subgroup of Π and we can introduce a strictly finer connected topological group topology on G using the epimorphism $p \restriction G^* : G^* \to G$. This completes the proof. □

Remark 4.4. One can generalize Theorem 4.1 substituting the condition that G is torsion-free by $|G/t(G)| \geq \mathfrak{c}$, where $t(G)$ denotes the torsion-subgroup of G. This generalization requires just the use of Corollary 3.3.

Theorem 4.5. *Let G be a connected separable Abelian torsion-free group. Then G admits a strictly finer connected separable topological group topology.*

Now we cannot directly apply the construction in the proof of Theorem 4.1, because the resulting dense subgroup G^* of $\Pi = G \times \mathbb{T}$ need not be separable (at least, the authors have not been able to prove that for a separable group G, *every* dense subgroup G^* of Π with $p(G^*) = G$ is separable). We need, therefore, two lemmas more. The first of them is a simple algebraic fact.

Lemma 4.6. *Let H be a countable subgroup of a torsion-free group G. There exists a countable pure subgroup L of G with $H \subseteq L$.*

Proof. For $n \in \mathbb{N}^+$ and $g \in G$, denote by $p(g,n)$ an element of G satisfying $np(g,n) = g$. If it does not exist, we simply put $p(g,n) = 0_G$. Define an increasing sequence $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ of countable subgroups of G in the following way. Put $H_0 = H$. If a countable subgroup H_k of G has been defined for some $k \in \mathbb{N}^+$, we put

$$X_k = H_k \cup \{p(g,n)\colon g \in H_k,\ n \in \mathbb{N}^+\} \quad \text{and} \quad H_{k+1} = \langle X_k \rangle.$$

It is clear that $L = \bigcup_{k \in \mathbb{N}} H_k$ is a countable subgroup of G satisfying $nG \cap L = nL$ for each $n \in \mathbb{N}^+$. Therefore, L is pure. □

Lemma 4.7. *Let G be an uncountable separable torsion-free topological group. There exist a countable subgroup K of G and a (discontinuous) homomorphism q of K to the circle group $\mathbb{T}$ such that the group $\{(x, q(x))\colon x \in K\}$ is dense in the product $\Pi = G \times \mathbb{T}$.*

Proof. Let S be a countable dense subset of G. Put $H = \langle S \rangle$. By Lemma 4.6 there is a countable pure subgroup L of G containing H. The quotient group G/L is obviously torsion-free. Denote by π the quotient homomorphism of G onto G/L and choose a countable independent subset $Y = \{y_n\colon n \in \mathbb{N}^+\}$ of the group $P = G/L$, that is, a set with the following property:

$$\text{if } k_1 y_1 + \cdots + k_n y_n = 0_P \text{ for some } n, k_1, \ldots, k_n \in \mathbb{N}^+, \text{ then } k_1 = \cdots = k_n = 0.$$

Let $\{t_n\colon n \in \mathbb{N}^+\}$ be a countable dense subset of $\mathbb{T}$. Define a function $f : Y \to \mathbb{T}$ by $f(y_n) = t_n$ for each $n \in \mathbb{N}^+$. Since, Y is an independent subset of P, f extends to a homomorphism $g : K_0 \to \mathbb{T}$, where $K_0 = \langle Y \rangle$. Put $K = \pi^{-1}(K_0)$ and $q = g\pi$. Obviously, K is a subgroup of G and $|K| = |K_0| \cdot |L| = \aleph_0$. It remains to show that the subgroup $\{(x, q(x))\colon x \in K\}$ is dense in Π.

Let $O = U \times V$ be a non-empty open subset of Π. There is an element $t_n \in V$. Choose $x_n \in K$ with $\pi(x_n) = y_n$. Then $q^{-1}(t_n) \supseteq \pi^{-1}(y_n) = x_n + L$. The set $x_n + L$ being dense in G, there exists $x \in (x_n + L) \cap U$. Thus, $x \in K$ and $(x, q(x)) = (x, t_n) \in O$. □

Proof of Theorem 4.5. Since G is non-trivial and connected, the cardinality of G is at least $\mathfrak{c}$. By Lemma 4.7, there exist a countable subgroup K of G and a homomorphism $q: K \to \mathbb{T}$ such that the set $D = \{(x, q(x)): x \in K\}$ is dense in $\Pi = G \times \mathbb{T}$.

Since $c(G) \le d(X) \le \aleph_0$ and $w(G) \le 2^{d(X)} = \mathfrak{c}$ (see [11, Theorem 2.3(i)] or [6, Theorem 1.5.6]), one can apply the construction used in the proof of Theorem 4.1 and define a homomorphism $h: G \to \mathbb{T}$ satisfying the following conditions:

(1) $h \restriction K = q$;

(2) the subgroup $G^* = \{(x, h(x)): x \in G\}$ of Π intersects every non-empty closed subset F of Π having the form $F = cl\, U \cap cl\, V$ for some open subsets U, V of Π and satisfying $Int\, p(F) \neq \emptyset$, where $p: \Pi \to G$ is the projection.

From (1) it follows that G^* contains a countable dense subgroup D, and hence is separable. From (2) and Proposition 4.2 it follows that G^* is connected. Since, the graph G^* of the homomorphism h is dense in Π, we conclude that h is discontinuous. Therefore, the group topology

$$\tau = \{p(O): O \text{ is open in } G^*\}$$

on G is strictly finer than the original topology of G. Clearly, the group (G, τ) is connected and separable. □

The following result illustrates an application of Theorem 4.1.

Corollary 4.8. *Let G be a connected dense torsion-free subgroup of a Cartesian product $\prod_{\alpha \in A} G_\alpha$, where each G_α is a separable group and $|A| \le \mathfrak{c}$. Then G admits a strictly finer connected topological group topology.*

Proof. Put $\Pi = \prod_{\alpha \in A} G_\alpha$. We have $w(G_\alpha) \le 2^{d(G_\alpha)}$ for each $\alpha \in A$ [6, Theorem 1.5.6], whence $w(\Pi) \le \mathfrak{c}$ and $w(G) \le w(\Pi) \le \mathfrak{c}$. Furthermore, since all the factors are separable, we conclude that $c(\Pi) \le \aleph_0$ [6, Corollary 2.3.18]. Being dense in Π, the group G is also ccc. The use of Theorem 4.1 completes the proof. □

We conclude with the following problems.

Problem 4.9. *Does a connected ccc Abelian torsion-free group admit a strictly finer connected topological group topology?*

Problem 4.10. *If G is a metrizable connected Abelian torsion-free group, does there exist a strictly finer connected topological group topology on G?*

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